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ENERGY-WAVE APPROACH IN THE MECHANICS OF BRITTLE DYNAMIC FRACTURE

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Almost all publications on the problems of brittle dynamic fracture mention that great difficulties are encountered in carrying out theoretical as well experimental studies. Analytical solutions of model problems are known only for finite regions and, practically speaking, the significance of these solutions has not yet been fully uncovered. Moreover, the search for analytical or numerical-analytical solutions for any law of crack distribution, even in infinite regions, is a "very complicated problem" [1]. In the given area of mechanics of a deformable solid, therefore, it is extremely important to develop new approaches, which perhaps are less general but "give more visible results" [1].

In this article we propose one such approach, based on the energy principle and the "time resolution" principle. On the basis of these principles we formulate the wave criterion of fracture, which can be used to solve specific practical problems of the mechanics of brittle dynamic fracture.

1. Formulation of the Problem. We consider a linearly elastic solid (isotropic or anisotropic, homogeneous or inhomogeneous) in an orthogonal (Cartesian) coordinate system {0, x, y, z}, containing defects such as cracks, voids, etc. (Fig. 1). As a result of external forces acting on the solid a major crack begins to propagate from the most loaded point of the solid (fracture source) at time $t = 0$, determined by the initiation (fracture) condition (criteria). The problem posed is that of estimating the dynamic growth of that crack.

2. Energy-Wave Approach. The main idea of the approach developed is that we do not look for an exact solution of this problem but, instead we propose to construct upper integral estimates of the dynamic crack growth. These integral estimates are constructed on the basis of the unperturbed propagation of the crack of the solution of the corresponding dynamic (or static) problem, the process for obtaining the latter being undoubtedly simpler.

We write the law of conservation of energy for a solid of finite dimensions. In the general case it has the form [2]

$$d(U + K) + dW = dA + dQ, \tag{2.1}$$

where U is the internal energy of low; K is the kinetic energy; and dW is the increment of fracture work necessary to form a new fracture surface. On the right is a general energy influx into the body because of the work of bulk and surface macroscopic forces dA and an external heat flux dQ.

Bulk macroscopic forces and the external influx dQ are assumed to be absent or so small as to be negligible. Integration of Eq. (2.1) over time in the range from $t = 0$ to $t = \tau$ for a material value Ω, bounded by the surface $\delta\Omega$ (Fig. 1), with allowance for the assumptions made leads to

$$(U + K) - (U^0 + K^0) + W = A. \tag{2.2}$$

Here U^0 and K^0 correspond to the initial values (the values at the initiation of the crack, i.e., at $t = 0$) of the internal and kinetic energy of the body, and U and K are the final values ($t = \tau$).

From Eq. (2.2) we easily obtain the upper estimate of the fracture work W. Noting that for an elastic medium $U + K \geq 0$, for W we have the inequality

$$W \leqslant L = A + (U^0 + K^0). \tag{2.3}$$

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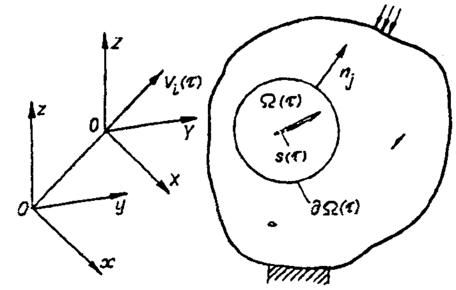

Fig. 1

Despite the significant simplifications, the last inequality written for an arbitrary material volume Ω cannot be used to obtain numerical estimates of the fracture work W since this requires knowledge of the work of the internal forces A on the surface $\partial\Omega$ of the material volume under consideration as a function of time, i.e., $A = A(\tau)$, which in the general case can be found only from the exact solution of the problem. But in this case all of simplifications made in order to obtain the integral estimates becomes meaningless.

If for Ω we choose the wave volume (region of influence) of the fracture processes the material volume of the body encompassed by perturbation from the fracture, i.e., from the new fracture surface forming at the time $t = \tau$, then the value of the external energy supply due to the work of surface macroscopic forces in this volume in the interval $[0\ldots\tau]$ is determined entirely by the solution corresponding to the problem with stationary defects, in other words a solution "not perturbed" by the crack growth. Indeed, points on the surface $\partial\Omega$ "do not know" about the onset of fracture in the interval from 0 to τ, since the perturbation from the crack propagation does not arrive at it, in accordance with the definition of the wave volume, until the time $t = \tau$ (we assume that the crack edges are free of external load).

With the introduction of the wave volume $\Omega(\tau)$ into the discussion we can achieve "time separation" (time decomposition) of the fracture processes and the supply of external energy to this volume. Thus, if $\Omega(\tau)$ is the wave volume, then to make estimates, according to Eq. (2.3), it is sufficient to know the solution of the corresponding problem divided that fracture does not occur, i.e., the crack does not propagate. The solution for the stationary crack condition (the crack can move but not propagate) can be obtained by well known and reliable analytical and numerical methods [3].

All of the discussions are also valid for any other volume encompassing the wave volume, but the accuracy of the estimates Eq. (2.3) can be shown to decrease in this case.

Returning to Eq. (2.3), for the total energy L of the wave volume $\Omega(\tau)$, we note that it is invariant under the choice of reference frame. Considering Eq. (2.3) relative to various inertial reference frames, moving relative to each other with nonzero velocity, we obtain a whole spectrum of upper estimates of the work of fracture, from which it is desirable to choose the smallest since it is closest to the real values of the work of fracture.

3. Wave Criterion of Fracture. Inequality (2.3) in component form can be written as

$$W(\tau) \leq L(\tau) = \frac{1}{2}\int\limits_{\Omega(\tau)} (\sigma^0_{ij}\varepsilon^0_{ij} + \rho v^0_i v^0_i)\,d\Omega + \int\limits_0^\tau \int\limits_{\partial\Omega(\tau)} \sigma_{ij} v_i n_j\, d\omega\, dt. \tag{3.1}$$

Here σ_{ij} and ε_{ij} are the stress and strain tensor; v_1 is the material particle velocity relative to the chosen reference frame; n_j is the unit vector normal to the outer surface $\delta\Omega$; an upper index 0 indicates that the value of the quantity is taken at the initial time, i.e., when the crack starts ($t = 0$); $\Omega(t)$ is the wave volume of the fracture process; τ is the running time measured from the onset of fracture; $t = [0\ldots\tau]$; $\partial\Omega(t)$ is the surface of the wave volume; and $d\Omega$ and $d\omega$ are elements of the volume and the surface $\Omega(\tau)$, respectively, over which the integration is carried out in Eq. (3.1). Clearly, the expression on the right side depends on the choice of reference frame, i.e., is not invariant under the Galilean transformation.

Considering Eq. (3.1) in another reference frame, moving with a nonzero velocity V_i relative to the first reference frame, we obtain a new, different estimate $W(\tau)$. It makes sense for each time τ to choose a reference system in which the upper estimate (3.1) is a minimum. The components of the displacement velocity of such a (new) reference frame $\{O, X, Y, Z\}$ relative to the old reference frame $\{O, x, y, z\}$ (see Fig. 1) is determined by

$$V_i(\tau) = \frac{1}{\int\limits_{\Omega(\tau)} \rho\, d\Omega}\left\{\int\limits_{\Omega(\tau)} \rho v^0_i\, d\Omega + \int\limits_0^\tau \int\limits_{\partial\Omega(\tau)} \sigma_{ij} n_j\, d\omega\, dt\right\}, \tag{3.2}$$

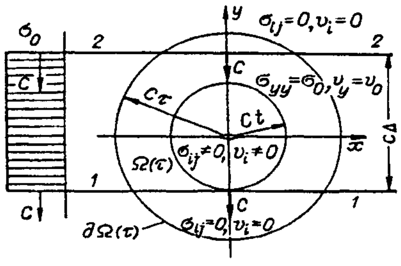

Fig. 2

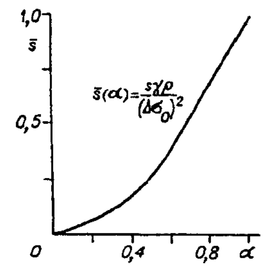

Fig. 3

obtained by finding the minimum of Eq. (3.1) as a function of the choice of reference frame.

Inequality (3.1) in the reference frame given by formula (3.2) and axes collinear to the old reference frame (see Fig. 1) is written as

$$W(\tau) \leq L(\tau) = \frac{1}{2}\int_{\Omega(t)} (\sigma^0_{ij}\varepsilon^0_{ij} + \rho(v^0_i - V_i(\tau))^2)d\Omega + \int_0^t \int_{\partial\Omega(\tau)} \sigma_{ij}(v_i - V_i(\tau))n_j d\omega dt. \tag{3.3}$$

Thus, Eqs. (3.2) and (3.3) give the most exact upper estimate, within the framework of the proposed treatment, of the work of fracture in an elastic body; henceforth we call it the wave criterion of fracture.

For practical applications it is important to know the direct growth of the crack (area of the fracture surface) s rather than the estimate of the work of fracture $W(\tau)$. For bodies subject to brittle fracture we can assume [2] that

$$dW = 2\gamma ds, \tag{3.4}$$

where γ is the specific work of fracture. It can depend on many parameters. Let us consider the case when $\gamma = \gamma(V)$, i.e., the specific work of fracture is a function of only the crack velocity V.

In the two-dimensional case s is given by two relations:

$$s = \int_0^t V dt; \tag{3.5}$$

$$2\int_0^{s(t)} \gamma(V)ds = W(\tau) \leq L(\tau). \tag{3.6}$$

By methods of variational calculations we can show that the functional (3.5) relative to $V = V(t)$ with the constraint (3.6) and satisfaction of the conditions

$$\frac{\partial^2}{\partial V^2}(\gamma(V)V) > 0 \tag{3.7}$$

has a maximum

$$s^+ = V^*\tau, \tag{3.8}$$

where V^* is the largest root of the algebraic equation

$$2\gamma(V)V = \frac{L(\tau)}{\tau}. \tag{3.9}$$

Thus, Eqs. (3.8) and (3.9) give the upper estimate of crack growth within the framework of linear fracture mechanics.

A review of the dynamic properties of various materials [4] shows that the condition (3.7) is almost always observed. For the particular case γ = const we have

$$s^+ = \frac{L(\tau)}{2\gamma}. \tag{3.10}$$

It is logical to call V^* the average velocity of crack propagation and $\gamma^* = \gamma(V^*)$, the specific dynamic work of fracture.

The wave criterion is easily generalized to the propagation of a finite number cracks and to the branching of a crack (or a combination of the two).

4. Corollaries of the Wave Criterion of Fracture. Let us consider the obvious corollaries of the wave criterion of fracture. If a body is in equilibrium and, therefore, v_i = const, then Eq. (3.3) transforms into the inequality

$$W(\tau) \leqslant L(\tau) = \frac{1}{2}\int_{\Omega(\tau)} (\sigma_{ij}\varepsilon_{ij})d\Omega. \tag{4.1}$$

The index 0 can be omitted, since the solution outside the wave volume at the initial time coincides with the solution $t > 0$.

We note that the "integral energy criterion of fracture," formulated by Ivanov [5] on the basis of an analysis of a large array of experimental data, reduces to Eq. (4.1). This criterion was used in [5] to estimate the fracture parameters of cylindrical shells under intense internal pressure but, as is seen from the discussions above, the application of Eq. (4.1) to such problems is not always correct.

Another corollary pertains to the case when the force terms of the wave criterion are small in comparison with the inertial terms, whereupon Eq. (3.3) reduces to

$$W(\tau) \leqslant L(\tau) = \frac{1}{2}\int_{\Omega(\tau)} (\rho v_i^0 v_i^0)d\Omega, \tag{4.2}$$

where v_i^0 was calculated relative to the reference frame bound to the center of inertia of the wave volume $\Omega(\tau)$ at $t = 0$. Such a criterion has been used to estimate the parameters of rapid pulsed fracture of composites.

For the practical use of the wave criterion of fracture and its corollaries (4.1) and (4.2) we must previously determine the direction of crack motion. It can be specified from the symmetry considerations of the problem or chosen as the most dangerous or undesirable.

5. Estimation of the Parameters of the Dynamic Fracture of Structures on the Basis of the Wave Criterion of Fracture. The procedure consists of several steps (stages).

1. The external and internal loads and various factors acting on the structure are determined.
2. The dynamic (static) problem is solved and the parameters of the stress–strain state of the structure are found without allowance for the propagation of defects.
3. The most loaded point of the body (source of fracture) and the displacement of the crack are determined in accordance with the conditions (criteria) of the initiation ($K \geq K_c$, $J \geq J_c, \ldots$).
4. The wave volumes for discrete times from formula (3.2) are found on the basis of the principles of the construction of the wave surfaces or rays [6].
5. The coordinate system is determined for chosen times in order to obtain the most exact upper estimate of the dynamic crack growth $s = s(\tau)$ within the framework of the treatment.
6. The direction of crack propagation is specified on the basis of the symmetry considerations or as the most dangerous and undesirable for the given structure.
7. The dynamic crack growth is estimated in accordance with the criterional relation (3.3) and Eqs. (3.8)-(3.10).

6. Example. We give an example of the dynamic growth of a crack in a homogeneous isotropic linearly elastic body in accordance with the treatment outlined above.

We consider the growth of a small crack under a square load pulse of length $\Delta(C\Delta \gg a$, the crack length) with stresses $\sigma_0(\sigma_0 > 0)$. The tensile stress σ_0 is assumed to be so large that the arrival of the leading edge *1* of the pulse at the crack tip is the criterion of crack initiation and the arrival of the trailing edge 2 is the criterion of crack arrest (Fig. 2).

Figure 2 shows the wave pattern that arises when a pulse impinges on a small crack. From it we see that the parameters of the stress–strain state on the surface of the wave volume $\partial\Omega$ in the interval from $t = 0$ to $t = \tau$ are either zero or are equal to the parameters of the incident pulse, i.e., $\sigma_{yy} = \sigma_0$, $v_y = v_0$, where $v_0 = \sigma_0/\rho C$ (we assume that $v_0 \ll C$); the total energy L of the wave volume $\Omega(\tau)$ thus does not depend on the fracture process. Knowing L, we can easily estimate the crack growth s from Eqs. (3.8)-(3.10).

Figure 3 shows the upper estimate of crack growth $\bar{s}$, $\bar{s}(\alpha) = s\gamma\rho/(\Delta\sigma_0)^2$ as a function of the dimensionless time $\alpha = \tau/\Delta$, obtained by substituting the values of the stress, strains, and velocities into the wave criterion of fracture (3.3) (it was assumed that $\gamma = \gamma(0)$ = const).

7. Limitations of the Energy-Wave Approach. The approach has obvious limitations in its application to problems of dynamic fracture, stemming from the arrival of the wave volume at the boundaries of the solid. For rigidly fastened or free boundaries the treatment can easily be generalized (the wave volume is constructed without these boundaries, since the energy flux through them is zero and, therefore, the total energy of the wave volume L remains (unchanged). The estimates are also valid if external energy is supplied to the body at a substantially slower rate than it is expended, i.e., much slower than the rate of energy absorption when a new fracture surface is formed, as is usually the case in brittle fracture, not to mention the case when the external energy flux is given.

In conclusion, we note that the area of possible application of the proposed energy-wave approach is limited by more than just the framework of fracture mechanics. Since the energy principle and the principle of time separation (time decomposition) are among the most general, the energy-wave approach will undoubtedly be useful in other areas of continuum mechanics, where energy is transformed or released at rapid rates.

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